

MAS212 Linear Algebra I 2007-2008

Coursework 6 — Solutions

Exercise* 1.

(a) Since

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \stackrel{R_3 - R_1}{=} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0,$$

the three vectors are linearly independent.

(b) Suppose that

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The corresponding system

$$\begin{array}{ccccccc} c_1 & & & & + & 3c_4 & = & 0 \\ & c_2 & + & c_3 & + & 2c_4 & = & 0 \\ c_1 & & & + & c_3 & + & c_4 & = & 0 \end{array}$$

is underdetermined, so it must have a nontrivial solution $(c_1, c_2, c_3, c_4)^T \neq \mathbf{0}$. Thus the four vectors are linearly dependent.

[Not required, but instructive, and certainly not forbidden to be looked at: solving the system above using Gaussian elimination the general solution is $c_4 = \alpha$, $c_3 = 2\alpha$, $c_2 = -4\alpha$, and $c_1 = -3\alpha$. So, for example choosing $\alpha = -1$, gives the linear dependence relation

$$3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . \quad]$$

(c) Since

$$\begin{vmatrix} 3 & -2 & 2 \\ 1 & 3 & 3 \\ 4 & 1 & 5 \end{vmatrix} \stackrel{R_1 + R_2}{=} \begin{vmatrix} 4 & 1 & 5 \\ 1 & 3 & 3 \\ 4 & 1 & 5 \end{vmatrix} \stackrel{R_1 - R_4}{=} \begin{vmatrix} 0 & 0 & 0 \\ 1 & 3 & 3 \\ 4 & 1 & 5 \end{vmatrix} = 0,$$

the three vectors are linearly dependent.

(d) Suppose that

$$c_1 \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The corresponding linear system is

$$\begin{array}{ccc} 3c_1 & + & 2c_2 & = & 0 \\ -3c_1 & & & = & 0 \\ c_1 & + & 2c_2 & = & 0 \end{array}$$

Equation 2 forces $c_1 = 0$. But then equation 1 (or 3) forces $c_2 = 0$, so the two vectors are linearly independent.

Exercise* 2.

- (a) False. The three vectors in $\mathbb{R}^{2 \times 2}$ are linearly dependent, because

$$3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 3 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This relation can be found by inspection, or, more systematically and also more laboriously, by writing down a system of three equations in three unknowns, which turns out to have a non-trivial solution.

- (b) True. In order to see this, suppose that

$$c_1 \mathbf{p}_1 + c_2 \mathbf{p}_2 + c_3 \mathbf{p}_3 = \mathbf{0},$$

so, for all t

$$c_1(1 - t^2) + c_2 t + c_3(t + 2t^2) = 0,$$

and thus, for all t

$$c_1 + (c_2 + c_3)t + (-c_1 + 2c_3)t^2 = 0.$$

The polynomial on the left-hand side of the above equation will be the zero polynomial if and only if all coefficients vanish, that is, the c_i must satisfy the following system

$$\begin{array}{rcl} c_1 & & = 0 \\ & c_2 + c_3 & = 0 \\ -c_1 & + 2c_3 & = 0 \end{array}$$

Now equation 1 forces $c_1 = 0$. But then equation 3 forces $c_3 = 0$, which finally forces $c_2 = 0$ using equation 2. Thus the three vectors are linearly independent.

Exercise* 3.

- (a) Since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a spanning set for V and $\mathbf{v}$ is in V , there are scalars $\alpha_1, \dots, \alpha_n$ such that

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n.$$

Thus

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n - \mathbf{v} = \mathbf{0},$$

so $\mathbf{0}$ can be written as a linear combination of $\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_n$, with not all coefficients vanishing (the coefficient of $\mathbf{v}$ is $-1 \neq 0$), so $\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly dependent.

- (b) By contradiction. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be linearly independent. Suppose to the contrary that $\{\mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a spanning set for V . Since $\mathbf{v}_1 \in V$, by (a), the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ must be linearly dependent, which is a contradiction. So the vectors $\mathbf{v}_2, \dots, \mathbf{v}_n$ cannot span V .

Exercise 4. Suppose that

$$c_1 \mathbf{y}_1 + \dots + c_n \mathbf{y}_n = \mathbf{0}.$$

Then

$$c_1 A \mathbf{x}_1 + \dots + c_n A \mathbf{x}_n = \mathbf{0},$$

so

$$A(c_1 \mathbf{x}_1 + \dots + c_n \mathbf{x}_n) = \mathbf{0}.$$

Since A is invertible, the above equation implies that

$$c_1 \mathbf{x}_1 + \dots + c_n \mathbf{x}_n = \mathbf{0}.$$

But since $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly independent, it follows that $c_1 = c_2 = \dots = c_n = 0$, so the vectors $\mathbf{y}_1, \dots, \mathbf{y}_n$ are linearly independent.

Exercise* 5.

- (a) The vectors are linearly independent, so we only need to check that they span $\mathbb{R}^3$. Thus, we need to show that, given an arbitrary vector $(a, b, c)^T \in \mathbb{R}^3$, we can find $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

However, we already saw in Exercise 1 (a) that the coefficient matrix of the corresponding 3×3 system is nonsingular, so it must have a solution $(\alpha_1, \alpha_2, \alpha_3)$ for any choice of a, b , and c . Thus the three vectors span $\mathbb{R}^3$, and hence they are a basis for $\mathbb{R}^3$, because they are also linearly independent.

- (b) Since the vectors are not linearly independent, they cannot form a basis for $\mathbb{R}^3$.
- (c) Again, these vectors are not linearly independent, so they cannot be a basis for $\mathbb{R}^3$.
- (d) The vectors are linearly independent. However, they do not form a basis for $\mathbb{R}^3$, since they do not span $\mathbb{R}^3$. In order to see this, note that if $(a, b, c)^T$ is an arbitrary vector in $\mathbb{R}^3$, then we cannot always find α_1 and α_2 such that

$$\alpha_1 \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

since the corresponding 3×2 system for α_1, α_2 is overdetermined.

[In fact, using Gaussian elimination it is easy to see that $(a, b, c)^T$ is in the span of the two vectors if and only if $3a + 2b - 3c = 0$.]

Exercise* 6. Since

$$\begin{pmatrix} r - 2s \\ r + s - t \\ s \\ t \end{pmatrix} = r \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{=\mathbf{x}_1} + s \underbrace{\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{=\mathbf{x}_2} + t \underbrace{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{=\mathbf{x}_3},$$

the vectors $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ span S . If they are linearly independent, they will form a basis for S . In order to check this suppose that

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The corresponding system is

$$\begin{array}{rclcl} c_1 & - & 2c_2 & & = & 0 \\ c_1 & + & c_2 & - & c_3 & = & 0 \\ & & c_2 & & & = & 0 \\ & & & & c_3 & = & 0 \end{array}$$

Equations 3 and 4 force $c_2 = c_3 = 0$, while equation 1 (or 2) then implies that $c_1 = 0$ as well. Thus $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent. Hence $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for S , so, in particular, $\dim S = 3$.

Exercise* 7. A 2×2 matrix is symmetric if and only if it is of the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$.

But

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{=E_1} + b \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{=E_2} + c \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{=E_3},$$

so $\text{Span}(E_1, E_2, E_3) = S$. Moreover, the vectors E_1, E_2, E_3 are linearly independent, since, if

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then

$$\begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so $c_1 = c_2 = c_3 = 0$. Thus $\{E_1, E_2, E_3\}$ is a basis for S , so, in particular, $\dim S = 3$.

Exercise 8. If $\mathbf{p} \in P_3$ has a zero at -1 , then $\mathbf{p}$ must be of the form $\mathbf{p}(t) = (t+1)(a+bt+ct^2)$, for some $a, b, c \in \mathbb{R}$. Thus

$$\mathbf{p}(t) = a(t+1) + b(t+1)t + c(t+1)t^2.$$

If we now define

$$\mathbf{p}_1(t) = (t+1), \quad \mathbf{p}_2(t) = (t+1)t, \quad \mathbf{p}_3(t) = (t+1)t^2,$$

then clearly $\text{Span}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = S$. In order to check that $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are linearly independent, suppose that

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0},$$

so, for all t

$$c_1(t+1) + c_2(t+1)t + c_3(t+1)t^2 = 0,$$

hence, for all t

$$c_1 + (c_1 + c_2)t + (c_2 + c_3)t^2 + c_3t^3 = 0.$$

Thus

$$\begin{array}{rcl} c_1 & & = 0 \\ c_1 + c_2 & & = 0 \\ & c_2 + c_3 & = 0 \\ & & c_3 = 0 \end{array}$$

which implies $c_1 = c_2 = c_3 = 0$. Thus $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a basis for S , and, in particular, $\dim S = 3$.